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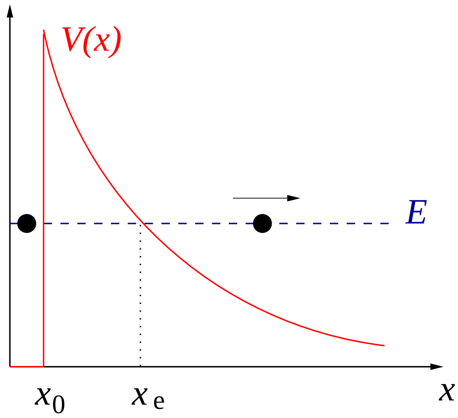
Problem Set 1 (due Monday, 19.10.2015 in the lecture)

(1.1) α-DECAY *(5 points)*

Consider an α-particle of mass m and charge $Z_\alpha e = 2e$, escaping from a nucleus whose charge drops down to Ze during the process. The potential is modelled as shown below. For $x < x_0$ the attractive nuclear force dominates, creating a narrow potential well, while for $x > x_0$ the Coulomb repulsion $Z_\alpha Z e^2/x$ takes over.



(i) The semiclassical mean life time τ has been introduced in the lecture: $\tau = 1/R$, $R = (v/2x_0)\mathrm{e}^{-\gamma}$, $v = \sqrt{2E/m}$, $\gamma = (2/\hbar)\int_{x_0}^{x_e} \sqrt{2m[V(x)-E]}\,\mathrm{d}x$. Show that for $x_e \gg x_0$ one obtains

$$\gamma \simeq \pi \frac{\sqrt{2m} Z_\alpha e^2}{\hbar} \left(\frac{Z}{\sqrt{E}} - \frac{4}{\pi} \frac{\sqrt{Z x_0}}{e\sqrt{Z_\alpha}} \right).$$

(ii) Using the above results, estimate the mean life time for an α-particle of kinetic energy 4.2 MeV, which tunnels out of a nucleus with final $Z = 90$. Assume $x_0 = 10^{-12}$ cm.

(iii)* [1]The decay law reads $N(t) = N(0)\mathrm{e}^{-t/\tau}$ so that the half-life time $T_{1/2} = (\ln 2)\tau = 0.693\,\tau$. Derive the TAAGEPERA-NURMIA formula

$$\log_{10} T_{1/2} = \alpha \left(\frac{Z}{\sqrt{E}} - Z^{2/3} \right) - \beta,$$

where $T_{1/2}$ is in years and E in MeV, and $\alpha \simeq 2$ and $\beta \simeq 30$ are constants.
Hint: Set $x_0 = \ell_0 Z^{1/3}$ (why?) with $\ell_0 \simeq 2 \times 10^{-15}$ m.

cont'd overleaf

[1] Starred problems (*) are voluntary but may bring you extra points.

(1.2) VARIATIONAL APPROACH *(2 points)*

Pretend you do not know the ground state wave function of the H atom (do you?). Determine an upper bound for the ground state energy by using $\psi(r) = \mathrm{e}^{-\alpha r^2}$ as a trial function.

(1.3) ERROR ESTIMATE FOR VARIATIONAL EIGENENERGY *(3 points)*

Let $\hat{H}|n\rangle = E_n|n\rangle$ and $E = \langle\psi|\hat{H}|\psi\rangle$, where the trial state $|\psi\rangle$ is already normalized, $\langle\psi|\psi\rangle = 1$. We define the *error vector*

$$|R\rangle = (\hat{H} - E)|\psi\rangle$$

and assume that E_k is that true eigenvalue of $\hat{H}$ which is closest to E. Prove that

$$E - \Delta \leq E_k \leq E + \Delta,$$

where $\Delta = \sqrt{\langle R|R\rangle}$.